Journal of Applied Mathematics and Mechanics 74 (2010) 144–153

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Singular boundary characteristics of the Hamilton–Jacobi equation☆

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ARTICLE INFO

Article history:
Received 3 March 2009

ABSTRACT

Situations exist in boundary value problems for first order partial differential equations arising in physics (the Hamilton–Jacobi equation), optimal control theory (the Bellman equation) and the theory of differential games (the Isaacs equation) when the value of the required function is not given on a part of the boundary or not at all, or it is not the limit of the (generalized) solution of the problem. Nevertheless, such conditions are required for constructing the solution (by the method of characteristics, for example). It is shown that the required boundary values can be exposed as a specific continuation of the conditions that are known in the boundary submanifolds of the given part of the boundary. This extension of the conditions is accomplished using the characteristic curves starting in a known submanifold of the boundary and running along the boundary. The characteristics are a generalization of the classical characteristics associated with a partial differential equation. They are called singular characteristics, and the theory of these has been developed in a number of the author's papers. After obtaining these "natural" boundary conditions, the solution is constructed using the conventional method of integrating the equations of the classical characteristics. Conditions of the Dirichlet and Neumann type are considered. The technique is illustrated using a numerical example from the theory of differential games containing a number of parameters.

1. Formulation of the problem

Consider a first order partial differential equation with the boundary conditions

$$F(x, u(x), p(x)) = 0, \quad x \in \Omega \subset R^n \ (p = \partial u/\partial x) \tag{1.1}$$

Generally speaking, the boundary conditions are only specif

$$u(x) = w(x), \quad x \in M \subset \partial\Omega \tag{1.2}$$

Generally speaking, the boundary conditions are only specified on a part M of the boundary $\partial\Omega$ of the (open) domain Ω. The boundary is conveniently represented in the form

$$\partial\Omega = M + M_0 \tag{1.3}$$

The lack of conditions on the subset M_0 is explained by different causes in different applications. For example, in the case of the generalized formulation of the problem,[1–6] the boundary values can be specified on the whole boundary $\partial\Omega$ and the set M is distinguished by the fact that, in this set, the solution is continuously coupled with the boundary value while, in the open subset M_0, the solution does not match to the boundary value. In optimal control and differential game problems, M_0 can arise as a phase constraint where the specification of the optimal result function is not assumed. In transform processing problems, the conditions can turn out to be fundamentally inaccessible.[7]

The continuous function $u(x)$, satisfying the boundary condition on M

$$u(x) = \omega(x), \quad x \in M$$

is called the generalized viscosity solution of problem (1.1), (1.2) and is simultaneously the upper and lower viscosity solution.

☆ *Prikl. Mat. Mekh.* Vol. 74, No. 2, pp. 202–215, 2010.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2010.05.003

For any trial function $\varphi(x)$, which is smooth in the neighbourhood of the point x_0 and such that a minimum

$$x_0\colon \min_x(u(x) - \varphi(x))$$

is reached at the point x_0, the upper solution satisfies the inequality

$$F(x_0, u(x_0), \nabla\varphi(x_0)) \geq 0, \quad x_0 \in \Omega + M_0 \tag{1.4}$$

and the lower solution satisfies the inequality

$$F(x_0, u(x_0), \nabla\varphi(x_0)) \leq 0, \quad x_0 \in \Omega \tag{1.5}$$

for any smooth trial function such that a maximum

$$x_0\colon \max_x(u(x) - \varphi(x))$$

is reached at the point x_0.

Note that the condition of the lower solution (1.5) is satisfied in the open domain Ω while the condition of the upper solution (1.4) must be satisfied up to the boundary of its subset M_0. This difference has been noted and proved[8] for the case of the Hamilton–Jacobi equation when the left-hand side of the equations has the special form:

$$F = p_n + H(x, u, \bar{p}); \quad \bar{p} = (p_1, \ldots, p_{n-1})$$

The variable separated out usually has the meaning of time: $x_n = t$, $t_0 \leq t \leq T$ and the boundary $\partial\Omega$ has an initial part M_i and a terminal part M_t. Moreover, the inclusions $M_i \subset M$, $M_t \subset M_0$ always hold.

The treatment in this paper is carried out in terms of the function $F(x, u, p)$ (an extended Hamiltonian). The Hamilton–Jacobi equation is encompassed here as a special case when $F = p_n + H$ (H is the Hamiltonian). The aim of this paper is to obtain the local necessary conditions for the singular boundary characteristics and to use them to derive the equations of the characteristics themselves.

2. The method of singular characteristics

It is well known that, for a definite smoothness of the functions $u(x)$, $F(x, u, p)$, the solution of problem (1.1) reduces to the integration of the system of (ordinary differential) characteristics of the equations

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p\rangle, \quad \dot{p} = -F_x - pF_u \ (p = \partial u/\partial x) \tag{2.1}$$

If the above mentioned functions are not smooth on a certain (smooth) manifold (surface), then it is necessary to invoke the so-called singular characteristics (SCs) for the constructions. The theory of SC has been proposed and developed earlier.[2,3,9] The differential-geometric treatment of regular characteristics (RCs) and SCs is the same[10]: The RCs determine the special tangential field in an even-dimensional ($2n$-dimensional) manifold W_1, defined in the $(2n+1)$-dimensional space of the variables (x, u, p) by the equation $F(x, u, p) = 0$ and the SCs determine the analogous field in the even-dimensional manifolds W_k of codimension $k = 1, 3, \ldots$. For the case $k = 3$, when the surface of the loss of smoothness of the functions $u(x)$ and/or $F(x, u, p)$ has dimension $n - 1$ in the space R^n, the manifold W_3 can be locally specified by the three equalities

$$W_3\colon F_1(x, u, p) = 0, \quad F_0(x, u, p) = 0, \quad F_{-1}(x, u, p) = 0 \tag{2.2}$$

The function F_i are determined by the type of singularity and, in this case, equalities (2.2) can represent the necessary conditions, having the meaning of the continuity of the generalized viscosity solution, the condition that the RCs we tangent to the singular surface and also of Eq. (1.1) itself.

In writing out the equations of the SCs, the singular Hamiltonian H^σ:

$$\mu H^\sigma = \{F_1F_0\}F_{-1} + \{F_0F_{-1}\}F_1 + \{F_{-1}F_1\}F_0 \tag{2.3}$$

is introduced into the treatment, where $\mu = \mu(x, u, p)$ is a non-zero normalizing factor and Jacobi (Poisson) brackets are denoted by the symbol $\{\cdot\}$:

$$\{FG\} = \langle F_x + pF_u, G_p\rangle - \{G_x + pG_u, F_p\}$$

The equations of the SC have the form of system (2.1), written in terms of the singular Hamiltonian and bounded in the manifold W_3:

$$\dot{x} = H_p^\sigma, \quad \dot{u} = \langle p, H_p^\sigma\rangle, \quad p = -H_x^\sigma - pH_u^\sigma \tag{2.4}$$

It has been shown[2,3,9] that the known types of singular surfaces arising in optimal control theory and the theory of differential games, that is, universal, equivocal and focal (in the terminology of Ref. 4), find their description in terms of SC (also, see Refs 11–13).

As an example, we will now present the basic relations of the theory of equivocal SCs. In an equivocal singular surface[4] Γ, the gradient $p(x)$ of the continuous solution undergoes a discontinuity; the RC approaches the surface from one side and departs from the other side undergoing a discontinuity. The surface Γ is characterized by the following three necessary optimality conditions defining a manifold W_3 of the form of (2.2)

$$F_0 = F = 0, \quad F_1(x,u) = u - \upsilon(x) = 0$$
$$F_{-1} = \{F_1 F\} = \langle F_p, p - q\rangle = 0 \left(q = \frac{\partial \upsilon}{\partial x}\right) \tag{2.5}$$

The last equality expresses by itself the condition that the departing RC touches the surface Γ and the vector p is the limit value of the gradient of the solution $u(x)$ as viewed from the departing RC. The function $\upsilon(x) \in C^2$ is the smooth branch of the solution of Eq. (1.1) along the other side of the surface Γ.

The algorithm of the SC method described above for the case of a smooth Hamiltonian $F \in C^2$ leads to the following system of equations with the homogeneity factor $\mu = \{\{F_1 F\} F_1\}$:

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p\rangle, \quad \dot{p} = -F_x - pF_u - \frac{\{\{FF_1\}F\}}{\{\{F_1F\}F_1\}}(p - q) \tag{2.6}$$

If the Hamiltonian F is not smooth and can be represented it in the form

$$F(x,u,p) = \min[H_0(x,u,p), H_1(x,u,p)] \tag{2.7}$$

where the functions H_i are smooth, then the manifold W_3 is described by the equalities

$$F_0 = H_0, \quad F_{-1} = H_1, \quad F_1(x,u) = u - \upsilon(x)$$

The corresponding system of SCs takes the form

$$\dot{x} = \lambda_0 H_{0p} + \lambda_1 H_{1p}, \quad \dot{u} = \lambda_0\langle p, H_{0o}\rangle + \lambda_1\langle p, H_{1p}\rangle$$
$$\dot{p} = -\lambda_0(H_{0x} + pH_{0u}) - \lambda_1(H_{1x} + pH_{1u}) - (\{H_1H_0\}/\mu)(p - q(x))$$
$$\lambda_0 + \lambda_1 = 1, \quad \lambda_0 = \{F_1H_1\}/\mu, \quad \mu = \{F_1H_1\} + \{H_0F_1\} \tag{2.8}$$

3. Singular characteristics on the boundary of a domain

3.1. Dirichlet conditions

We will represent the part of the boundary M_0 in equality (1.3) in the form of the sum:

$$M_0 = M_0^+ + M_0^-$$

The surface M_0^+ is characterized by the fact that the RCs approach (possibly, with contact) M_0^+ from internal points of the domain Ω and they depart (possibly, with contact) from points of the surface M_0^- within the domain Ω.

We will assume that the gradient $p = u_x$ of the solution $u(x)$ extends continuously from the domain Ω to the surface M_0^-. Suppose the domain Ω is locally representable in the neighbourhood of M_0^- in the form

$$\Omega = \{x \in R^n | g(x) < 0\} \tag{3.1}$$

where $g(x)$ is a smooth function. Then, the part M_0^- of the boundary ∂Ω is defined by the equality $g(x) = 0$. By definition, the inequality

$$M_0^-: \langle F_p, g_x\rangle \le 0 \tag{3.2}$$

holds at the points of the submanifold M_0^- since the vector g_x is an outer normal to the boundary ∂Ω at the point x.

We shall initially consider the case of a smooth Hamiltonian $F(x, u, p)$. It can be shown that, in the case of a family of trial functions with a scalar parameter λ

$$\varphi(x,\lambda) = u(x) + \lambda g(x), \quad \lambda \ge 0$$

a local minimum in the difference $u(x) - \varphi(x,\lambda) = -\lambda g(x)$ is reached at the boundary point for which $g(x) = 0$. Condition (1.4) for the upper solution must be satisfies up to the boundary $M_0^- \subset \Omega$. Then, the relations

$$f(\lambda) = F(x,u,p + \lambda g_x) \ge 0, \quad \lambda \ge 0, \quad f(0) = 0 \tag{3.3}$$

hold for the above mentioned family of trial functions.

The condition $f(\lambda) \ge 0$ may be violated either in the neighbourhood of zero or for finite values of λ. This requirement in the neighbourhood of zero is equivalent to the condition of a one-sided local minimum of the function $f(\lambda)$ at the point $\lambda = 0$. We will now formulate these conditions in the form of a lemma.

Lemma. Suppose the inequality $f(\lambda) \ge 0$ holds when $\lambda \in [0, \lambda_0]$ for a point $x \in M_0^-$. For the derivative of the function $f(\lambda)$ when $\lambda = 0$, we then have

$$f(0) = \langle F_p, g_x \rangle \geq 0$$

If this derivative turns out to be equal to zero:

$$\langle F_p, g_x \rangle = \{gF\} = 0 \tag{3.4}$$

then the following equality for the second derivative is satisfied

$$f'(0) = \langle F_{pp} g_x \rangle = -\{\{gF\}g\} \geq 0 \tag{3.5}$$

Since the part M_0^- of the boundary of the domain is considered, comparison with (3.2) shows that equality (3.4) holds and, thereby, inequality (3.5). Furthermore, we will assume that the required condition $f(\lambda) \geq 0$ is satisfied for all remaining λ outside the neighbourhood of zero. If this condition is violated, the trial function $u(x)$ cannot be a generalized solution of Eq. (1.1) in the neighbourhood of a point $x \in M_0^-$.

Hence, three relations, defining a certain manifold W_3, are satisfied on a part of the boundary M:

$$F_0 = F = 0, \quad F_1(x) = g(x) = 0, \quad F_{-1} = \{gF\} = \langle F_p, g_x \rangle = 0 \tag{3.6}$$

Using a singular Hamiltonian of the form of (2.3), it is possible to write the equations of the SCs (2.4) that, in the case of the functions (3.6), give the following equations which are similar to system (2.6):

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p \rangle, \quad \dot{p} = -F_x - pF_u + \frac{\{\{Fg\}F\}}{\langle F_{pp} g_x, g_x \rangle} g_x \tag{3.7}$$

Here, the homogeneity factor also has the form

$$\mu = \{\{F_1 F\} F_1\} = \{\{gF\}g\} = -\langle F_{pp} g_x, g_x \rangle$$

Note that, generally speaking, both the numerator and the denominator of the fraction, representing the coefficient of g_x, are positive. The sign of the denominator determines condition (3.5), and the inequality $\{\{Fg\}F\} \geq 0$ is a sufficient condition for the existence of a solution of the irregular Cauchy problem with a tangential departure of the characteristics on the boundary (see Ref. 3).

In the special case, when $g(x) = -x_n$, that is, the boundary of the domain is locally the coordinate plane $x_n = 0$, the contact condition (3.6) gives $F_{p_n} = 0$, and system (3.7) in the coordinate-wise notation takes the form

$$\begin{aligned} &\dot{x}_k = F_{p_k}, \quad \dot{u} = \langle p, F_p \rangle, \quad \dot{p}_k = -F_{x_k} - p_k F_u, \quad k = 1, \ldots, n-1 \\ &\dot{x}_n = F_{p_n} = 0, \quad \dot{p}_n = -F_{x_n} - p_n F_u + \{F_{p_n} F\}/F_{p_n p_n} \end{aligned} \tag{3.8}$$

In the case of a Hamiltonian of the form of (2.7) which is not smooth, the conditions $F_i = 0$, defining the manifold W_3, have the form

$$W_3\colon F_0 = H_0, \quad F_1(x) = g(x), \quad F_{-1} = H_1$$

This leads to a system of SCs that is similar to system (2.8) but differs from it in that $p - q(x)$ is replaced by g_x and F_1 is replaced by g (we call this System A). Here, the regular characteristics depart from M_0^- transversely. In principle, it can turn out that the non-smoothness of the Hamiltonian is not manifested on the domain boundary and the situation is similar to the case of a smooth Hamiltonian. Then, the SCs have the form of (3.7) in terms of one of the smooth branches H_0 or H_1 and the RCs depart tangentially. System (3.7) has been obtained earlier.[14] This system had been derived even earlier [15] for an optimal control problem with a phase constraint when the contact condition (the last chain of equalities in (3.6)) follows from the optimality condition.

4. Singular characteristics on the boundary of a domain. The Neumann conditions

Suppose the domain Ω is locally defined by the inequality

$$\Omega = \{x \in R^n | g(x) < 0\} \tag{4.1}$$

where $g(x)$ is a smooth scalar function with a non-zero gradient. The boundary $\partial\Omega$ is then defined by the equality $g(x) = 0$. Suppose a boundary condition of the Neumann type

$$\partial u/\partial\gamma = \langle \gamma(x), \nabla u(x) \rangle = 0$$

is specified on $\partial\Omega$. Here, $\gamma = \gamma(x)$ is a smooth vector field defined on the boundary $\partial\Omega$. Then, the following three equalities (2.2) of the form $F_i(x, u(x), p(x)) = 0$

$$F_0(x, u, p) = F(x, u, p), \quad F_1(x, u, p) = g(x), \quad F_{-1}(x, u, p) = \langle p, \gamma(x) \rangle$$

hold in $\partial\Omega$ and define the manifold W_3 of co-dimension 3 in the space of (x, u, p).

Hence, the manifold W_3, in the case of the Neumann type conditions considered, is determined directly from the data of the problem and there is no need to derive additional necessary conditions for using the SC method. A Neumann-type condition can be considered as a special case of the more general specification of the boundary conditions in the form of the equality

$$F_{-1}(x, u(x), p(x)) = 0, \quad x \in \partial\Omega$$

that also contains the gradient p of the required function u. Here, F_{-1} is a certain known (given) function.

These three equalities generate the following system of SC that is analogous to System A:

$$\dot{x} = H_p = \lambda_0 F_p + \lambda_1 \gamma(x), \quad \dot{u} = \langle p, H_p\rangle = \lambda_0\langle p, F_p\rangle + \lambda_1\langle p, \gamma(x)\rangle = \lambda_0\langle p, F_p\rangle$$

$$\dot{p} = -H_x = -\lambda_0(Fx + pFu) - \lambda_1\gamma_x p - ([\langle F_x, \gamma(x)\rangle - \langle F_p, \gamma_x p\rangle]/\mu)\nabla g$$

$$\lambda_0 + \lambda_1 = 1, \quad \lambda_0 = \{F_1 g\}/\mu = -\langle g, \gamma\rangle/\mu$$

$$\mu = \langle g_x, F_p\rangle - \langle g_x, \gamma\rangle = \langle g_x, F_p - \gamma\rangle$$

Here, as in System A, a normalization condition of the form $\lambda_0 + \lambda_1 = 1$ is used which corresponds to a specific choice of the time scale. In each specific problem, another more suitable normalization condition can be chosen. For example, the requirement that $\lambda_0 = 1$ leads to the equality $\mu = -\langle g_x, \gamma\rangle$.

In the special case when the latter coordinate plane serves as the boundary $\partial\Omega$, $g(x) = x_n = 0$, $\gamma(x)$ is a constant vector field and the corresponding system of SCs takes the form (in the case of the normalization condition $\mu = \gamma_n$)

$$\dot{x} = F_p - (F_{p_n}/\gamma_n)\gamma, \quad \dot{u} = \langle p, F_p\rangle$$

$$\dot{p}_i = -F_{x_i}, \quad i = 1, \dots, n-1, \quad \dot{p}_n = -F_{x_n} + \langle F_x, \gamma(x)\rangle\gamma_n \tag{4.2}$$

The first equation reduces to $\dot{x}_n = 0$ which is true for any motion along the boundary. Furthermore, if the function F is independent of u, then the equations of the SC can be written in the form

$$\dot{x} = RF_p(x, p), \quad \dot{p} = -R^T F_x(x, p); \quad R = I - N\gamma^T/\langle\gamma, N\rangle$$

where R is the reflection matrix and N is the unit normal to the boundary of the domain. The equations for the boundary extremals were presented earlier in this form.[16] Putting $\gamma = N = (0, \dots, 0, 1)$ in the system of SCs, it can be simplified to the following form

$$\dot{x}_i = F_{p_i}, \quad \dot{x}_n = 0, \quad \dot{u} = \langle p, F_p\rangle, \quad \dot{p}_i = -F_{x_i}, \quad i = 1, \dots, n-1, \quad \dot{p}_n = 0 \tag{4.3}$$

For the last coordinate we have $x_n = 0$, $p_n = 0$, as follows from the specification of the boundary itself and the conditions on the boundary: $\partial u/\partial N = \partial u/\partial x_n = p_n = 0$.

Although the equations for x_i and p_i are identical in form to the equations of the RC, this system represents the SC because of the equations for the last coordinate. The equalities $\dot{x}_n = F_{p_n}$, $\dot{p}_n = -F_{x_n}$ hold for the RC starting on the boundary and, generally speaking, $F_{x_n} \neq 0$.

The SC method generalizes the results in Ref. 16 to the case of a curvilinear boundary.

5. Example

We will now consider the following boundary-value problem in the function $u(t,x)$ in a domain Ω of the plane (t, x), $t, x \in R$.

$$F(t, x, p, q) = q + \sqrt{a^2 + p^2} + t\sqrt{b^2 + p^2} = 0, \quad (x, t) \in \Omega$$

$$u(0, x) = kx, \quad x \geq 0$$

$$\Omega = \{(x, t): t > 0, -x + \alpha t < 0\}$$

$$p = \partial u/\partial x, \quad q = \partial u/\partial t, \quad a, b, k, \alpha = \text{const}, \quad b > a > 0, \quad k < 0 \tag{5.1}$$

Boundary conditions are specified on the semi-axis $x \geq 0$, which is a part of the boundary $\partial\Omega$ of the domain Ω (Fig. 1). The open half-line (the lateral boundary of the domain):

$$M_0 = \{(\alpha, t): \ t > 0, \quad g(x, t) = -x + \alpha t = 0\}.$$

is the part M_0 on which the boundary values of the required function $u(t, x)$ are not given. In Fig. 1, this half line passes through the point t_0 when $\alpha > 0$, the points t_0 and t_1 when $\alpha = 0$ and the points t_2 and t_1 when $-1 < \alpha < 0$.

Equation (5.1) has been considered earlier.[3] It can be shown that it is the Bellman–Isaacs equation for the following differential game with a fixed terminal time

$$x' = dx/dt = V_1 + tU_1, \quad U_1^2 + U_2^2 \leq 1, \quad V_1^2 + V_2^2 \leq 1$$

$$J = \int_{\tau_0}^{\theta}(aV_2 + tbU_2)d\tau + kx(\theta) \to \min_U \max_V, \quad t = \theta - \tau$$

The terminal set is the semi-axis $x \geq 0$, $\tau = 0$ $(t = 0)$ and a phase constraint of the form $-x + \alpha t \leq 0$ is imposed.

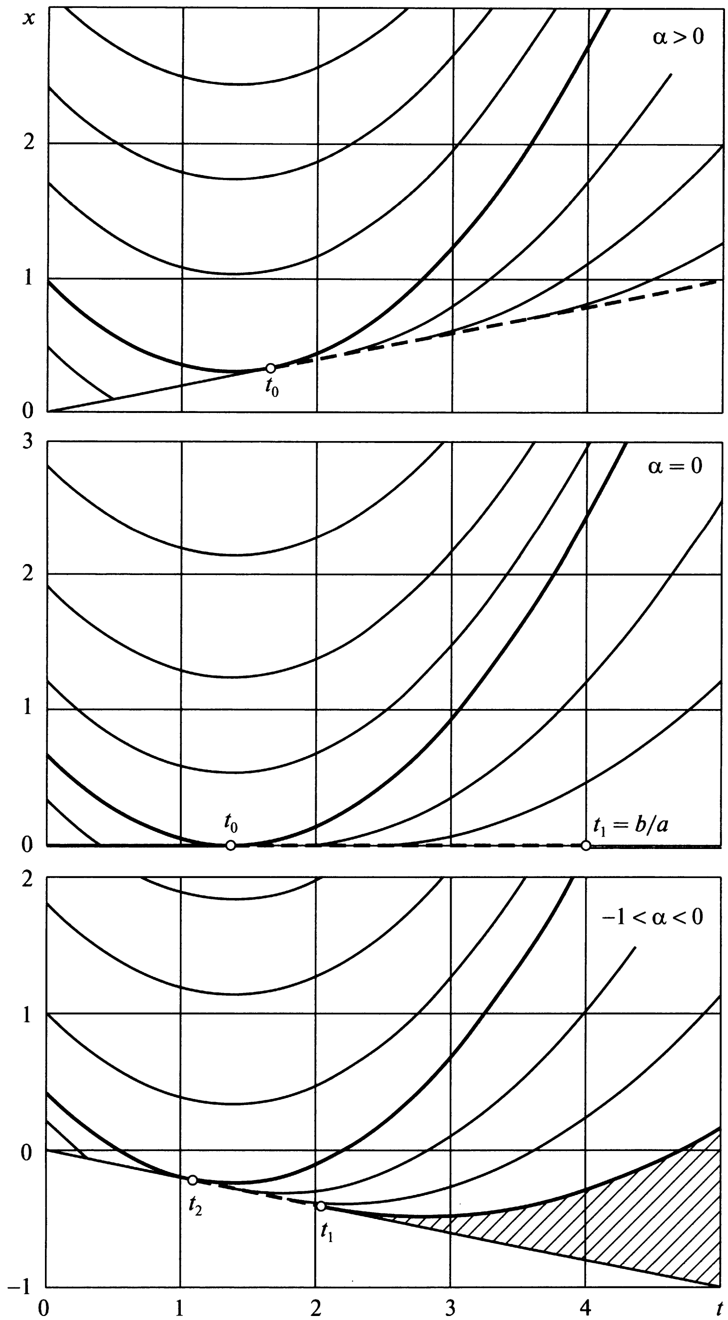

Fig. 1.

We emphasize that, in this paper, problem (5.1) is solved on the basis of the necessary conditions for SCs (and RCs) for first order partial differential equations and the game treatment of the problem is not considered. The equations of the RCs (2.1), are written, together with the corresponding initial data, in the form

$$\dot{x} = F_p = H_p = p/\sqrt{a^2+p^2} - tp/\sqrt{b^2+p^2}$$
$$\dot{u} = pF_p + qF_q = pF_p + H, \quad \dot{p} = -F_x = 0$$
$$x(0) = s, \quad u(0) = ks, \quad p(0) = k, \quad s \geq 0$$
$$H = H(t,p) = \sqrt{a^2+p^2} - t\sqrt{b^2+p^2} \tag{5.2}$$

Instead of five equations, three are written here since the equation $\dot{t} = F_q = 1$ for the variable t is omitted and the variable q can be found from the condition

$$F = q + H(t,p) = 0$$

Integrating Eqs (5.2) with the given initial conditions, it is possible, firstly, to find that parabolae of the family

$$x - C = \frac{kt}{\sqrt{a^2 + k^2}} - \frac{kt^2}{2\sqrt{b^2 + k^2}} \tag{5.3}$$

are the characteristics (the projections of the integral lines on the (t, x) plane) and, secondly, to obtain the following continuous solution (the primary solution) of Eq. (5.1)

$$u(t, x) = kx - t\sqrt{a^2 + k^2} + \frac{t^2}{2}\sqrt{b^2 + k^2} \tag{5.4}$$

which is defined in a certain part of the domain Ω. A parabola of the family (5.3) determines this part and, depending on the parameters of the problem, this parabola either touches the lateral boundary of the domain Ω at the point (see Fig. 1)

$$t_0 = L(k) - \alpha N(k); \quad L(k) = \sqrt{\frac{b^2 + k^2}{a^2 + k^2}}; \quad N(k) = \frac{\sqrt{b^2 + k^2}}{k} \tag{5.5}$$

while the parameter of the parabola has the value

$$C = \alpha t_0 - \frac{kt_0}{\sqrt{a^2 + k^2}} + \frac{kt_0^2}{2\sqrt{b^2 + k^2}}$$

or it passes through the origin of coordinates, and, then, $C = 0$. Formula (5.5) is found by equating the derivative $\dot{x}$ on the characteristic (5.3) and on the straight line

$$-x + \alpha t = 0 \tag{5.6}$$

Solution (5.4) is not defined on the right of this parabola. The segment of the line (5.6) from the origin of the coordinates to the point $t = t_0$ is the part M_0^+ of the submanifold of M_0 where the RCs departing the domain Ω, are coming. The ray on this line, where $t \geq 0$, is the part of M_0^- where the boundary conditions to be extended by using the SCs.

The function $g(x, t)$ for the problem considered and the condition for contact on the boundary $g(x, t) = 0$ have the form

$$g(x, t) = -x + \alpha t, \quad \{gF\} = -H_p + \alpha = 0$$

Direct calculations for the repeated Jacobi brackets give

$$\{\{Fg\}F\} = -\frac{p}{\sqrt{b^2 + p^2}}, \quad \{\{gF\}g\} = -H_{pp} \tag{5.7}$$

The equations of the SCs, (3.7) and (3.8), then take the form

$$\dot{x} = H_p(=\alpha), \quad \dot{p} = \frac{p}{H_{pp}\sqrt{b^2 + p^2}}, \quad \dot{u} = p\alpha - H \tag{5.8}$$

Integration of system (5.8) enables us to obtain the value of the solution $u(x, t)$ on part of the lateral side (5.6). This part can be a segment of finite (possibly zero) length, corresponding to a time interval $[t_0, t_1]$, $t_0 \leq t_1$, or an infinite ray ($t_1 = \infty$). After this, the equations of the RC with the boundary conditions can be integrated in the above mentioned part of the line (5.6). Generally speaking, the instant t_1 is determined by the condition that the denominator in the equation for the conjugate variable, that is for the normalization factor $\mu = \{\{gF\}g\}$, vanishes (see (2.6) and (3.7)). It is obvious from relations (5.7) that, in the problem considered, this condition is equivalent to the second derivative of the Hamiltonian vanishing: $H_{pp} = 0$.

Because of the fact that the problem considered is two-dimensional, only the last of the three equations (5.8) is subject to integration since the relation $x(t)$ is given by the equation of the boundary (5.6), and the relation $p(t)$ (or $t(p)$) can be found from the contact condition $H_p - \alpha = 0$. Using the expression for H_p (5.2), we obtain

$$t = L(p) - \alpha N(p) \tag{5.9}$$

Note that formula (5.3) represents the relation (5.9) at the specific point $p = k$, the point of origin of the SC. The relations $t = t(p)$ are shown in Fig. 2 for different values of the parameter α. An analysis of the curves in Fig. 2 noticeably simplifies the constructions in this problem.

Only negative values of p occur on the SCs. This follows from the condition $\{\{Fg\}F \geq 0\}$ and the formula for the given Jacobi bracket presented above. We recall that this condition is necessary in order that the solution of the irregular Cauchy problem, in which the characteristics depart from the boundary with contact, is determined on the necessary side of the boundary. In other words, in order that the characteristics depart toward the interior of the domain Ω and not its exterior. This condition was obtained in Ref. 3 and subsequently presented in a simpler formulation in Ref. 17.

We will now describe of the constructions depending on the value of the parameter d.

The case when $\alpha > 0$. It can be seen from formula (5.5) and Fig. 2 that, for any $k < 0$, $t_0 > 0$ holds and a SC exists which goes from the point t_0 to infinity. This enables us to fill the whole of the domain Ω, as shown in the upper part of Fig. 1.

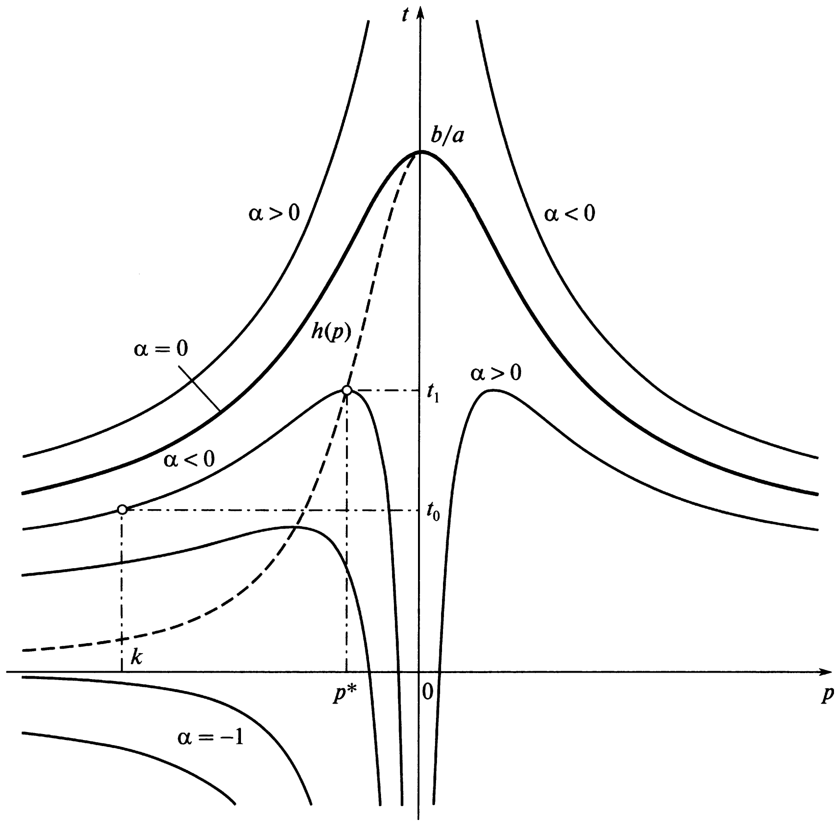

Fig. 2.

The case when $\alpha = 0$. A special case is obtained here which leads to system (3.8). By reversing relation (5.3), it is possible to obtain

$$p = -a\sqrt{p((b/a)^2 - t^2)/(t^2 - 1)}$$

When the constraints $b > a$ and $k < 0$ are imposed on the parameters, the relation $p(t)$ is defined in the time interval $1 \le t \le t_1$, $t_1 = b/a$ but the SC exists in the interval $[t_0, t_1]$. The instant $t_1 = b/a$ corresponds to the point of termination of the SC. It is easy to show that $1 < t_0 < t_1$. From the points of the interval $[t_0, t_1]$ on the time axis, it is necessary to produce RCs which also turn out to be parabolae and become progressively flattened as the right end of the interval is approached. When $t = t_1 = b/a$, the parabola degenerates into a ray $t \ge t_1$ by which filling of the whole of the domain considered by the characteristics is ensured (the middle part of Fig. 1). The solution is calculated by integrating the last equation of (5.8) along the characteristics. These constructions give a smooth continuation of the function (5.4) over the whole of the domain Ω.

The case when $\alpha < 0$. Here, depending on the values of the parameters k and α, a SC may or may not exist. If it exists, the instant t_1 of its termination corresponds to the point of a local maximum of the relation $t(p)$. In order to show this, we will analyse the qualitative behaviour of the curves in Fig. 2.

By definition, the relation $t(p)$ satisfies the equality

$$H_p(t(p), p) - \alpha = 0$$

identically with respect to p. Differentiating it with respect to p, we conclude that the point $p^* = p^*(\alpha)$ of the extremum (maximum), $t'(p^*) = 0$, corresponds to the singularity $H_{pp} = 0$.

We will now show that the condition $t(p^*) > 0$ is always satisfied at the extremum point. To do this, we consider the curve $t = h(p)$, that is, the geometric locus of the maximum points of the relations $t = t(p) = t(p, \alpha)$ in the (p, t) plane (see Fig. 2). The maximum point of the relation $t = t(p, \alpha)$ is found from the conditions

$$t = t(p) = L(p) - \alpha N(p), \quad t'(p) = L'(p) - \alpha N'(p) = 0$$

The second equation gives $\alpha = L(p)/N(p)$. Substituting this expression into the first equation, it is possible to obtain the following representation for the function $t = h(p)$

$$h(p) = L(p)^2 \frac{\tilde{N}'(p)}{N'(p)} > 0; \quad \tilde{N}(p) = \sqrt{a^2 + p^2}\, p$$

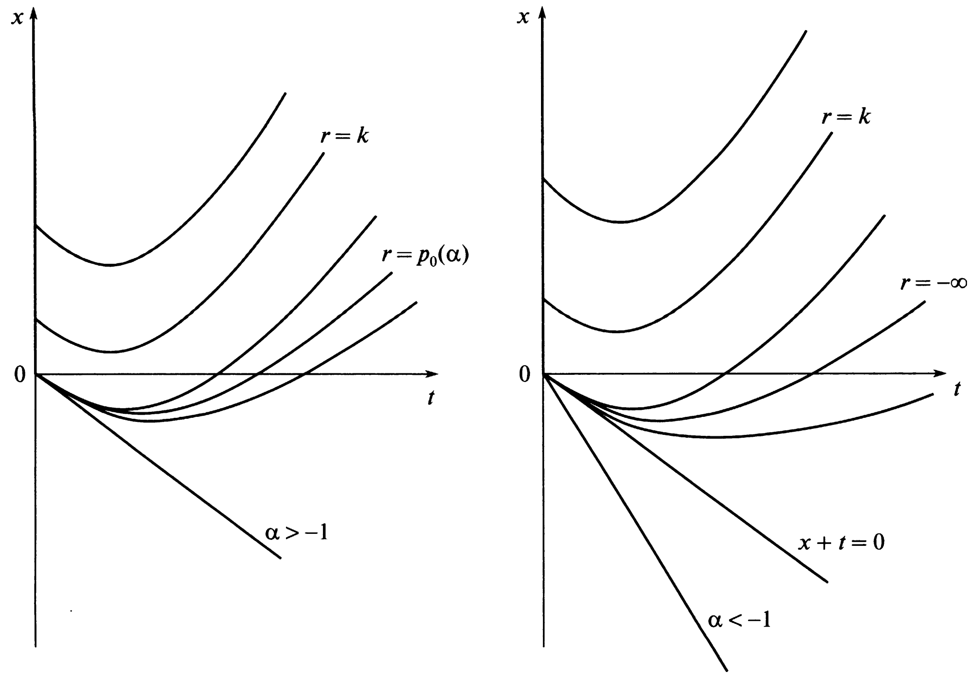

Fig. 3.

The strict inequality here follows from the strict monotonicity of the function $\tilde{N}(p)$ in the interval $p \in (-\infty, 0)$ for any value of the parameter $\alpha > 0$. Moreover, it can be shown that the function $h(p)$ is also strictly monotonic and

$$h(-\infty) = a/b < 1, \quad h(0) = b/a > 1$$

Since $\alpha = L'(p)/N(p) \to -1$ when $p \to -1$, the relation $t = t(p, \alpha)$ only has a maximum with respect to p for $-1 < \alpha < 0$.

The curve $t = h(p)$ in the (p, t) plane separates the regions where $H_{pp} > 0$ (above the curve in the left half-plane $p < 0$) and where $H_{pp} < 0$ (below the curve). The necessary condition for the existence of a SC

$$-\{\{gF\}g\} = H_{pp} \geq 0$$

is not satisfied in the region below the curve (see relations (3.5) and (5.6)).

The above analysis enables us to distinguish four subcases with different behaviour of the RC and SC.

Subcase 1: $-1 < \alpha < 0$, $k < p^*(\alpha)$. A SC exists in the interval $[t_0, t_1]$, $0 < t_0 = t(k) < (p^*) = t_1$. It can be seen from Fig. 2 that the relation $t(p)$ along the SC exists in the interval $[t_0, t_1]$ and increases monotonically with respect to p. In order to construct the solution $u(t, x)$, it is necessary to produce RCs from points of the straight line (5.6) in the interval $[t_0, t_1]$. To the right of the latter, the characteristics are parabolae starting from the point t_1 and an unfilled subdomain remains, shown hatched in the lower part of Fig. 1.

Subcase 2: $-1 < \alpha < 0$, $p^*(\alpha) < k < p_0(\alpha)$. The value of p_0 satisfies the condition $t(p_0) = 0$ and can be found using the equality (5.9): $p_0(\alpha) = \alpha a/\sqrt{1-\alpha^2}$. Here, there is no SCs, it is impossible to pick out a monotonic branch of $t(p)$ and the condition $H_{pp} > 0$ is violated. A positive instant t_0 (5.7) exists when the parabola touches the lateral boundary of the domain. To the right of this parabola, an unfilled subdomain remains, which corresponds to the lower part of Fig. 1, if we put $t_1 = t_0$.

Subcase 3: $-1 < \alpha < 0$, $p_0(\alpha) \leq k < 0$. As in the preceding subcase, there is also no SC here. The family of characteristics (5.3) does not touch the lateral side with the exception that there is possibly contact at the origin of the coordinates. In the case of the strict inequality $p_0(\alpha) < k$, a family of characteristics (an integral funnel) of the form (5.3) when $C = 0$:

$$x = \frac{rt}{\sqrt{a^2 + r^2}} - \frac{rt^2}{2\sqrt{b^2 + r^2}}, \quad p_0(\alpha) \leq r \leq k$$

can be produced from the origin of the coordinates as well as from a corner point of the boundary. The parabola, corresponding to $r = p_0(\alpha)$, touches the lateral side at the origin of the coordinates, and an unfilled subdomain remains to the right of it. Such a pattern is presented in the left part of Fig. 3.

Subcase 4: $\alpha \leq -1$, $k < 0$. For the above mentioned reasons, there is also no SCs here and an integral funnel exists which starts from the origin of coordinates and corresponds to the interval $-\infty < r \leq k$ of the parameter r. The last parabola of the family, corresponding to $r = 1$, does not touch the lateral side and, at the origin of coordinates, has a tangent $\dot{x} = H_p = -1 \geq \alpha$. This is the smallest possible value of H_p in this problem. An unfilled subdomain also remains to the right of this parabola. The corresponding pattern is presented on the right-hand part of Fig. 3.

Note that the function $u(t, x)$, obtained with the given constructions for Subcases 3 and 4, is differentiable everywhere, apart from the origin of the coordinates $t = 0$, $x = 0$. A smooth approximation to the function $u(t, x)$ and to the pattern of the characteristics can be obtained by smoothing out the angle on the boundary of the domain at the origin of coordinates.

It can be seen from the plots that a subdomain remains for a negative value of α, which is not filled with characteristics. The existence of this denotes that the initial boundary value problem for the Hamilton–Jacobi–Bellman–Isaacs equation can have a non-unique solution. Indeed, by specifying different boundary values on the part of the straight line (5.6) free from singular characteristics, it is possible to obtain different solutions of the problem.

In the case of a differential game, this fact means that, in starting from the above mentioned unfilled part of the domain Ω, it is impossible to bring a phase point onto the semi-axis $t=0$, $x\geq 0$ without violating the constraint $-x+\alpha t\leq 0$.

The solution of the example that has been described is based on the equations of the boundary SC that are obtained in the form of necessary conditions characterizing the generalized solution of a partial differential equation. For complete verification of the solution, it is additionally necessary to verify condition (1.4), or condition (3.3) that replaces it, on the part of the boundary considered. In the given example, by virtue of the fact that grad $g=(-1, \alpha)$, condition (3.3) takes the form

$$f(\lambda) = F(t, x, p-\lambda; q+\lambda\alpha) = q+\lambda\alpha+\sqrt{a^2+(p-\lambda)^2}-t\sqrt{b^2+(p-\lambda)^2}\geq 0$$

where $\lambda\geq 0$.

In conclusion, we will compare the internal and boundary singular characteristics in this problem. Equation (5.1) has been considered[3] with non-smooth boundary conditions that generate a non-smooth solution. If the parameters a and b are subject to the condition $a>b$, a singular equivocal surface (line) arises in the problem, described by system (2.6), in which there is a jump in the gradient. If the function (5.4) is taken as the initial solution $\upsilon(t, x)$, then the contact condition (2.5) is written in the form

$$0 = \{u-\upsilon(t, x), F\} = \left(p-\frac{\partial\upsilon}{\partial t}\right)H_p+q-\frac{\partial\upsilon}{\partial t} = (p-k)H_p+H(t, k)-H(t, p)$$

and the equations of the equivocal singular line take the form

$$\dot{x} = H_p, \quad \dot{p} = \frac{\sqrt{b^2+k^2}-(b^2+kp)/\sqrt{b^2+p^2}}{(p-k)H_{pp}}, \quad \dot{u} = pH_p-H$$

These equations have a structure similar to system (5.8), but somewhat more complicated than it.

Acknowledgement

This research was financed by the Russian Foundation for Basic Research (07-01-00418).

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Translated by E.L.S.